1 Preliminaries

Let $\mathbb{N}$ be the set of non-negative natural numbers. A smooth map is a map of class C^∞.

Definition 1.1 (Chart)**.** Let $(X, \mathcal{T})$ be a topological space and $n \in \mathbb{N}$. An *n-dimensional chart on* $(X, \mathcal{T})$ is a pair (U, φ), where $U \in \mathcal{T}$ and φ is a homeomorphism such that $\varphi[U] \subset \mathbb{R}^n$ is open.

Definition 1.2 (Locally Euclidean)**.** A topological space $(X, \mathcal{T})$ is said to be *locally Euclidean* if there exists some $n \in \mathbb{N}$ such that for all $x \in X$ there exists an n-dimensional chart (U_x, φ_x) with $x \in U$. The pair (U_x, φ_x) is called a *chart at* $x \in X$. If (U_1, φ_1) and (U_2, φ_2) are two n-dimensional charts on $(X, \mathcal{T})$ such that $U_1 \cap U_2 \neq \emptyset$, then

$$\varphi_1 \circ \varphi_2^{-1} : \varphi_2[U_1 \cap U_2] \longrightarrow \varphi_1[U_1 \cap U_2] \quad \text{and} \quad \varphi_2 \circ \varphi_1^{-1} : \varphi_1[U_1 \cap U_2] \longrightarrow \varphi_2[U_1 \cap U_2]$$

are homeomorphisms between open subsets of $\mathbb{R}^n$, called the *overlap functions*.

Definition 1.3 (Topological manifold)**.** A topological space $(X, \mathcal{T})$ is a *topological manifold* if $(X, \mathcal{T})$ is Hausdorff, locally Euclidean and second countable.

Definition 1.4 (Atlas)**.** Let $(X, \mathcal{T})$ be a topological manifold. An *n-dimensional atlas on* $(X, \mathcal{T})$ is a collection $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of n-dimensional charts on $(X, \mathcal{T})$, for some indexing set I, such that

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(U_\alpha \cap U_\beta) \quad \text{and} \quad \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are diffeomorphisms for all $\alpha, \beta \in I$, and $\bigcup_{\alpha \in I} U_\alpha = X$. A chart (U, φ) on $(X, \mathcal{T})$ is said to be *admissible* to $\mathcal{A}$ if

$$\varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \longrightarrow \varphi(U \cap U_\alpha) \quad \text{and} \quad \varphi_\alpha \circ \varphi^{-1} : \varphi(U \cap U_\alpha) \longrightarrow \varphi_\alpha(U \cap U_\alpha)$$

are diffeomorphisms for all $\alpha \in I$, and $\mathcal{A}$ is said to be *maximal* if for every admissible chart (U, φ) there exists some $\alpha \in I$ such that $U = U_\alpha$ and $\varphi = \varphi_\alpha$. A maximal n-dimensional atlas $\mathcal{A}_{\max} = \{(V_\alpha, \psi_\alpha)\}_{\alpha \in I}$ on $(X, \mathcal{T})$ is called a *differentiable structure on* $(X, \mathcal{T})$

Definition 1.5 (Differentiable manifold)**.** A *differentiable manifold* is a triple $(X, \mathcal{T}, \mathcal{A})$, where $(X, \mathcal{T})$ is a topological manifold and $\mathcal{A}$ is a differentiable structure on $(X, \mathcal{T})$.

Definition 1.6 (Topological group)**.** A group $(G, \cdot)$ is a *topological group* if G is also a Hausdorff space and the maps

$$\begin{aligned} \mathcal{M}_G : G \times G &\longrightarrow G \\ (g_1, g_2) &\longmapsto g_1 \cdot g_2 \end{aligned} \tag{1}$$

and

$$\begin{aligned} \mathrm{Inv}_G : G &\longrightarrow G \\ g &\longmapsto g^{-1} \end{aligned} \tag{2}$$

are continuous.

Definition 1.7 (Lie group)**.** A group $(G, \cdot)$ is a *Lie group* if G is also a differentiable manifold and $\mathcal{M}_G$ and Inv_G are smooth.

Definition 1.8 (Tangent space)**.** Let $(X, \mathcal{T})$ be a differentiable manifold and $x \in X$. A *tangent vector to* X *at* x is a map $\mathbf{v} \in (C^\infty(X))^*$ for which the *Leibniz Product Rule* holds, i.e.

$$\mathbf{v}(fg) = f(p)\mathbf{v}(g) + \mathbf{v}(f)g(p) \tag{3}$$

for all $f, g \in C^\infty(X)$. The set $T_pX := \{\mathbf{v} \in (C^\infty(X))^* : \mathbf{v} \text{ is a tangent vector to } X \text{ at } x\}$ is called the *tangent space to* X *at* x, and has a natural vector space structure where the operations are defined as

$$(\mathbf{v} + \mathbf{w})(f) = \mathbf{v}(f) + \mathbf{w}(g) \quad \text{and} \quad (a\mathbf{v})(f) = a\mathbf{v}(f) \tag{4}$$

for all $a \in \mathbb{R}$ and for all $f, g \in C^\infty(X)$.